Thin Position and Essential Planar Surfaces

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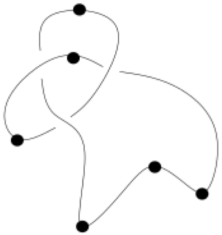
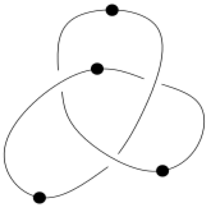
Outline

- Background and definitions
  - Width
  - Thin and thick levels
  - Thin position and bridge position
  - Essential surfaces
- Results
  - Main theorem
  - Corollary and discussion

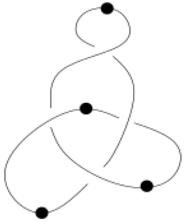
Width



D_1

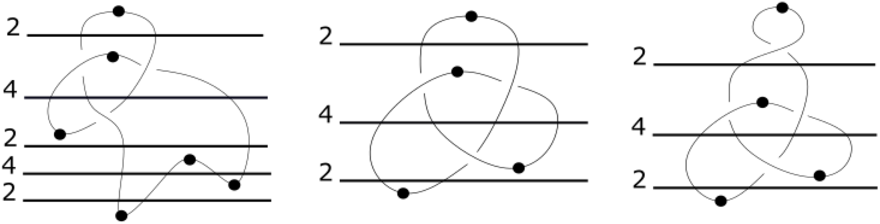


D_2

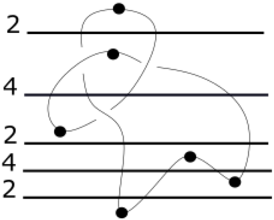


D_3

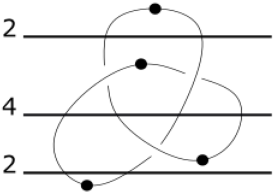
Width of knot diagram



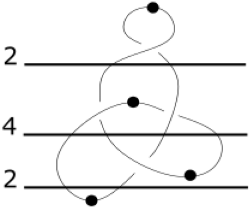
Width of knot diagram



$W(D_1) = 14$

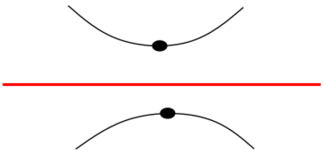


$W(D_2) = 8$

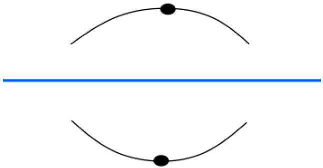


$W(D_3) = 8$

Thin and thick levels



Thin level



Thick level

Thin position and bridge position

Thin position: Diagram minimizing width
Bridge position: Diagram with no thin level

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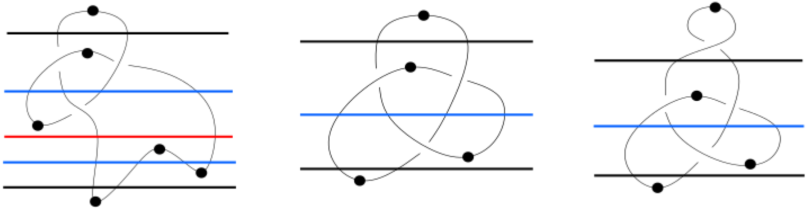
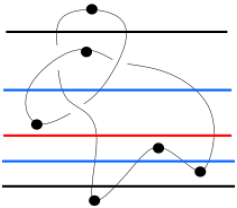
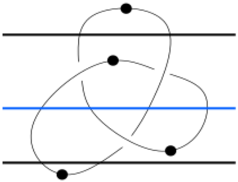


Thin position and bridge position

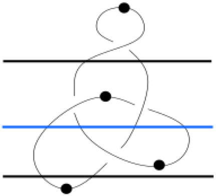
Thin position: Diagram minimizing width
Bridge position: Diagram with no thin level



Not thin position
Not bridge position

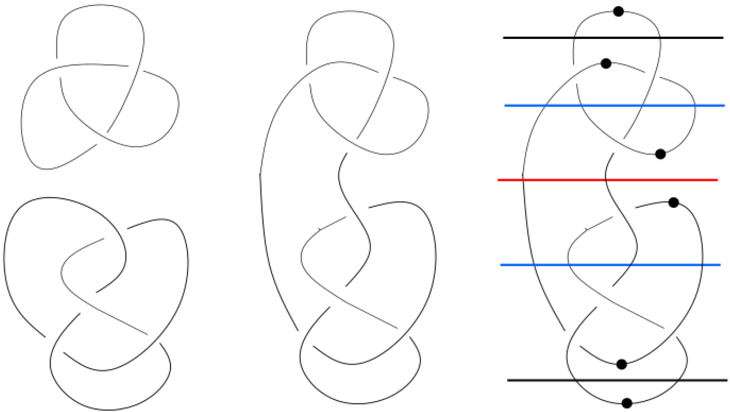


Thin position
Bridge position



Thin position
Bridge position

Thin position and not bridge position



Level surfaces

Embed knots in $S^3 = \mathbb{R}^3 \cup \{\infty\}$.
Lower dimension: $S^2 = \mathbb{R}^2 \cup \{\infty\}$.

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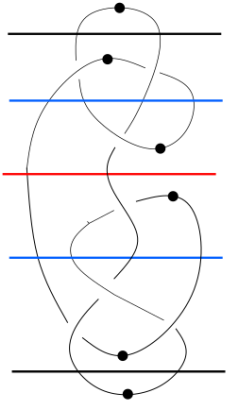
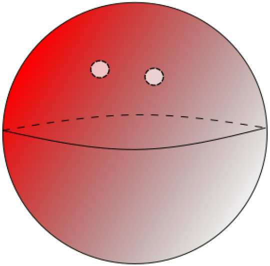


Knot diagram



Level surface

Essential surfaces

Essential surface: a properly embedded, orientable surface in a 3-manifold M is **essential** in M if

- it is incompressible
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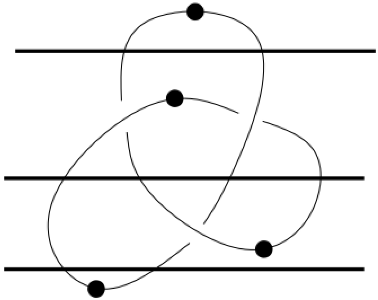

Compressible vs. incompressible

Compression disk: Given a surface S embedded in a 3-manifold M, a disk D embedded in M is a compression disk of S if

- $D \cap S = \partial D$
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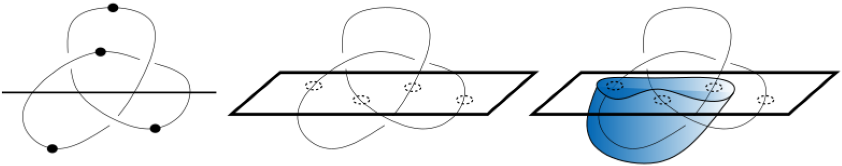
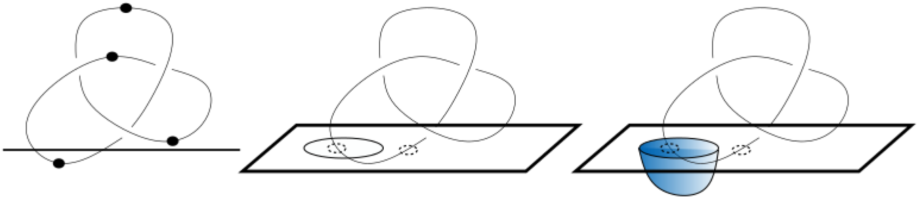
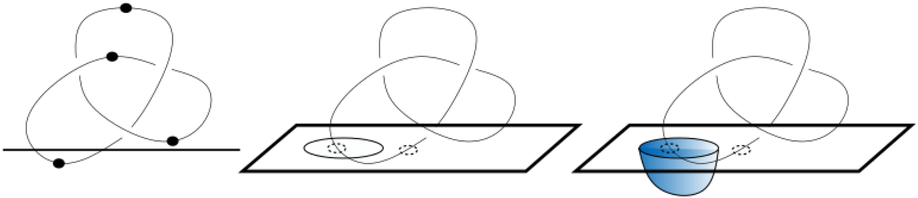


Incompressible level surface

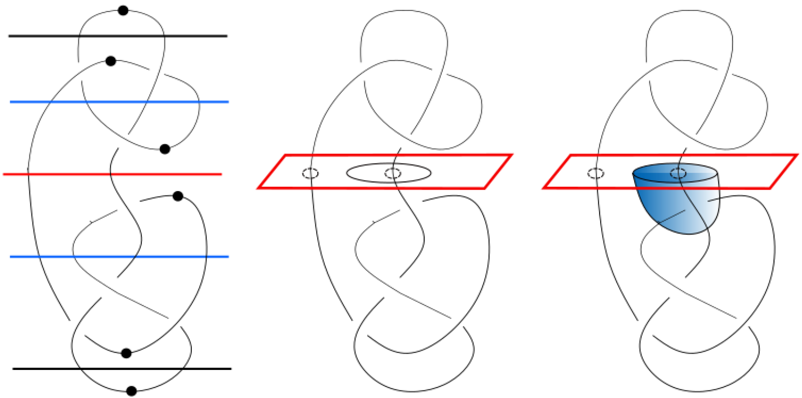


Incompressible level surface



Incompressible, but boundary parallel, so not essential

Essential level surface



Main Result

Theorem (Wu, 2006)

If a knot K in S^3 is in thin position but not in bridge position, then a thinnest level surface of K is an essential surface in $S^3 \setminus K$.

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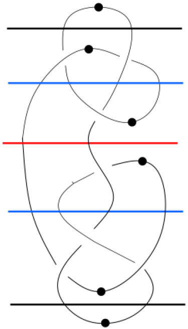
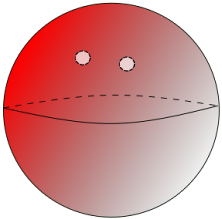
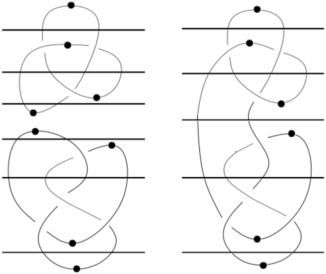




Discussion and Corollary

We know that $W(K_1 \# K_2) \leq W(K_1) + W(K_2) - 2$ for all knots K_1 and K_2.



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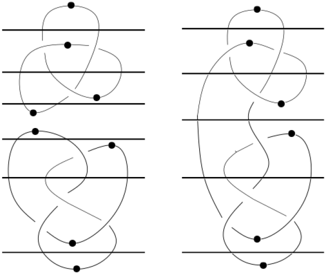

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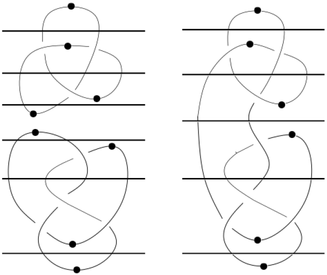

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Theorem (Rieck and Sedgwick, 2002)

The conjecture holds for small knots. (Corollary to Wu's theorem.)

Discussion

In 2010, Ryan Blair and Maggy Tomova found a counterexample to the conjecture that $W(K_1 \# K_2) = W(K_1) + W(K_2) - 2$ for all knots K_1 and K_2.

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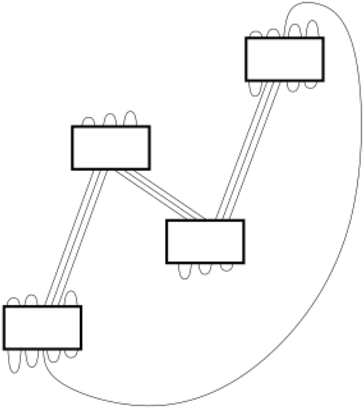

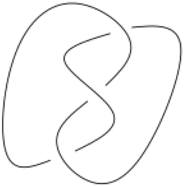

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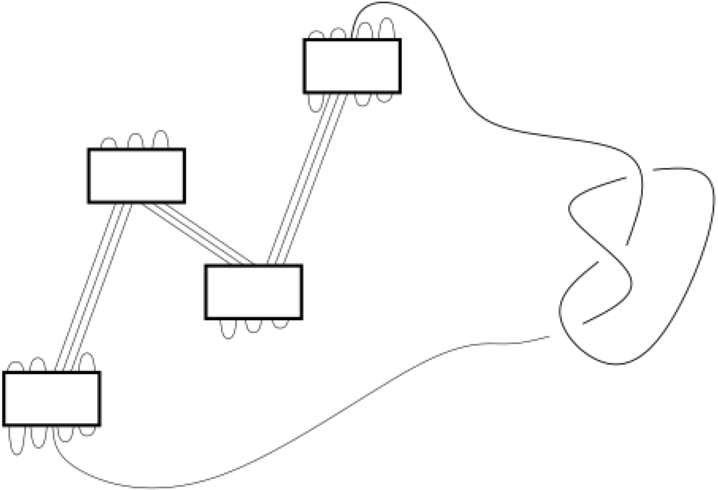

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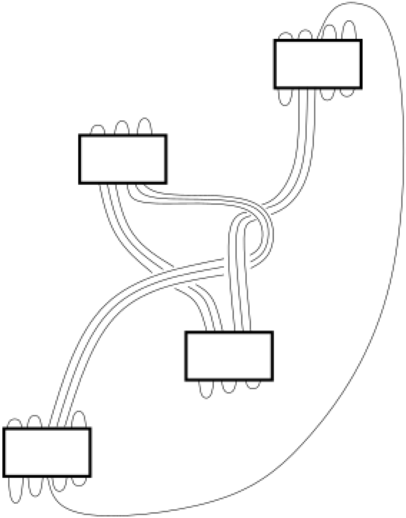

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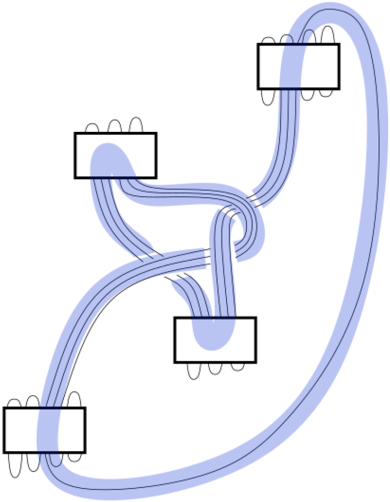

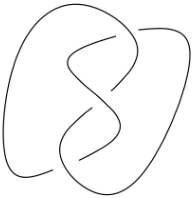

References

Ryan Blair, Maggy Tomova
Width is Not Additive
(2010)

Ying-Qing Wu
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(2006)